B. Sc. (Hons.) Part III

7th Paper

17.07.2020

<u>Standard form (2) continued</u>

Q. 1 Find the solution of the equation

$$z = px + qy - 2\sqrt{pq}.$$

Soln The given equation

$$z = px + qy - 2\sqrt{pq} \quad \text{---} (1)$$

It is of the form

$$z = px + qy + f(p, q).$$

Its complete integral is given by

$$z = ax + by - 2\sqrt{ab} \quad \text{---} (2)$$

Put $b = \phi(a)$

$$\Rightarrow z = ax + y\,\phi(a) - 2\sqrt{a\,\phi(a)} \quad \text{---} (3)$$

Differentiating it partially with respect to a, we get

$$0 = x + y\,\phi'(a) - 2 \times \frac{1}{2\sqrt{a\,\phi(a)}} \times \left[\phi(a) + a\phi'(a)\right] \quad \text{---} (4)$$

Elimination of a from (3) and (4) gives the general integral of the given differential equation.

Now, we find its singular integral.

From (2)

$$z = ax + by - 2\sqrt{ab} \quad \text{---} (2)$$

Differentiating it partially w.r.t to a, we have

$$0 = x - \frac{\cancel{2}}{\cancel{2}\sqrt{ab}} \times b \Rightarrow x = \sqrt{\frac{b}{a}} \quad \text{---} (3)$$

Differentiating (2) partially w.r.t to b, we get

$$0 = 0 + y - \frac{\cancel{2}}{\cancel{2}\sqrt{ab}} \times a$$

$$\Rightarrow y = \sqrt{\frac{a}{b}} \quad \text{---} (4)$$

Now,

$$x - z = x - (ax + by - 2\sqrt{ab}) \quad [\text{using } (2)]$$

$$= \sqrt{\frac{b}{a}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$$

$$= \sqrt{\frac{b}{a}} - \sqrt{ab} - \sqrt{ab} + 2\sqrt{ab}$$

$$\Rightarrow x - z = \sqrt{\frac{b}{a}} \quad \text{---} (5)$$

Again

$$y - z = y - (ax + by - 2\sqrt{ab}) \quad [\text{using } (2)]$$

$$= \sqrt{\frac{a}{b}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$$

$$= \sqrt{\frac{a}{b}} - \sqrt{ab} - \sqrt{ab} + 2\sqrt{ab}$$

$$= \sqrt{\frac{a}{b}} \quad \text{---} (6)$$

From (5) and (6) $(x-z)(y-z) = 1$ This is the singular integral.

2. Prove that the complete integral of the differential equation

$$z = px + qy + \frac{pq}{pq - p - q}$$ represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.

Soln The given equation

$$z = px + qy + \frac{pq}{pq - p - q} \quad \text{--- (1)}$$

This is of the form

$$z = px + qy + f(p, q)$$

Its complete integral is given by

$$z = ax + by + \frac{ab}{ab - a - b}, \quad \text{--- (2)}$$

where a and b are constants.

$$\Rightarrow ax + by - z = \frac{ab}{a + b - ab}$$

This is a linear equation in x, y and z.

So, eq (2) represents planes.

Rewriting (2), we have

$$ax + by - z = \frac{ab}{a+b-ab}$$

$$\Rightarrow \frac{ax}{\frac{ab}{a+b-ab}} + \frac{by}{\frac{ab}{a+b-ab}} - \frac{z}{\frac{ab}{a+b-ab}} = 1$$

$$\Rightarrow \frac{x}{\frac{b}{a+b-ab}} + \frac{y}{\frac{a}{a+b-ab}} + \frac{z}{\frac{ab}{ab-a-b}} = 1$$

So, the intercepts on the x, y and z-axes are $\frac{b}{a+b-ab}$, $\frac{a}{a+b-ab}$, $\frac{ab}{ab-a-b}$ respectively.

$$\text{Their sum} = \frac{b}{a+b-ab} + \frac{a}{a+b-ab} + \frac{ab}{ab-a-b}$$

$$= \frac{a+b-ab}{a+b-ab} = 1.$$

$\Rightarrow$ algebraic sum of the intercepts on three coordinates axes $= \underline{\underline{1}}$